MINIMOGs and Mathematical blackjack

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Abstract

This is an exposition of some ideas of Conway, Curtis, and Ryba on $S(5,6,12)$ and a card game called mathematical blackjack.

Contents

An *m-(sub)set* is a (sub)set with m elements. For integers $k < m < n$, a *Steiner system* $S(k,m,n)$ is an n-set X and a set S of m-subsets having the property that any k-subset of X is contained in exactly one m-set in S. For example, if $X = \{1,2,...,12\}$, a Steiner system $S(5,6,12)$ is a set of 6-sets, called *hexads*, with the property that any set of 5 elements of X is contained in ("can be completed to") exactly one hexad.

This note focuses on $S(5,6,12)$. If S is a Steiner system of type $(5,6,12)$ in a 12-set X then the symmetric group S_X of X sends S

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to another Steiner system $\sigma(S)$ of X. It is known that if S and S' are any two Steiner systems of type $(5,6,12)$ in X then there is a $\sigma \in S_X$ such that $S' = \sigma(S)$. In other words, a Steiner system of this type is unique up to relabelings. (This also implies that if one defines M_{12} to be the stabilizer of a fixed Steiner system of type $(5,6,12)$ in $X = \{1,2,...,12\}$ then any two such groups, for different Steiner systems in X, must be conjugate in S_X. In particular, such a definition is well-defined up to isomorphism.)

1 Curtis' kitten

J. Conway and R. Curtis [Cu1] found a relatively simple and elegant way to construct hexads in a particular Steiner system $S(5,6,12)$ using the arithmetical geometry of the projective line over the finite field with 11 elements. This section describes this.

Let

$$\mathbf{P^1}(\mathbf{F_{11}}) = \{\infty, \mathbf{0}, \mathbf{1}, \mathbf{2}, ..., \mathbf{9}, \mathbf{10}\}$$

denote the projective line over the finite field $\mathbf{F_{11}}$ with 11 elements. Let

$$Q = \{0,1,3,4,5,9\}$$

denote the quadratic residues and 0 and let

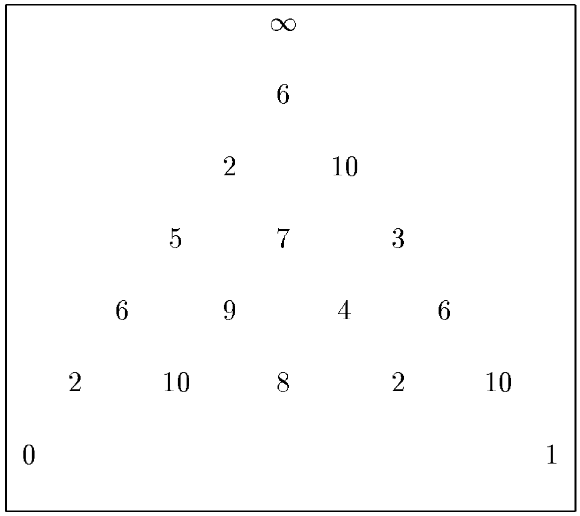
$$L = < \alpha, \beta > \cong PSL(2, \mathbf{F_{11}}),$$

where $\alpha(y) = y+1$ and $\beta(y) = -1/y$. Let

$$S = \{\lambda(Q) \mid \lambda \in L\}.$$

Lemma 1 *S is a Steiner system of type* $(5,6,12)$.

The elements of S are known as *hexads* (in the "modulo 11 labeling").



Curtis' Kitten.

The "views" from each of the three "points at infinity" is given in the following tables.

6	10	3
2	7	4
5	9	8

picture at ∞

5	7	3
6	9	4
2	10	8

picture at 0

5	7	3
9	4	6
8	2	10

picture at 1

Each of these 3×3 arrays may be regarded as the plane $\mathbf{F}_3^2$. The *lines* of this plane are described by one of the following patterns.

•	•	•
×	×	×
○	○	○

slope 0

•	×	○
•	×	○
•	×	○

slope infinity

•	×	○
○	•	×
×	○	•

slope -1

×	○	•
○	•	×
•	×	○

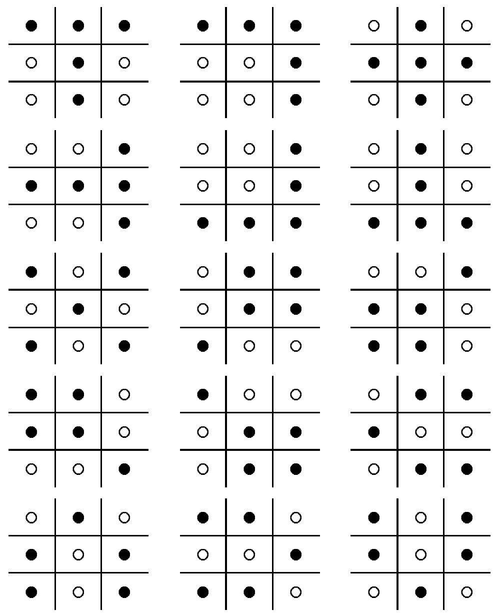
slope 1

The union of any two perpendicular lines is called a *cross*. There are 18 crosses. The *crosses* of this plane are described by one of the following patterns of filled circles.

•	•	•
•	○	○
•	○	○

•	○	○
•	•	•
•	○	○

•	○	○
•	○	○
•	•	•



The complement of a cross in $\mathbf{F}_3^2$ is called a *square*. Of course there are also 18 squares. The *crosses* of this plane are described by one of the above patterns of hollow circles.

The hexads are

1. $\{0, 1, \infty\} \cup \{\text{any line}\}$,
2. the union of any two (distinct) parallel lines in the same picture,
3. one "point at infinity" union a cross in the corresponding picture,
4. two "points at infinity" union a square in the picture corresponding to the omitted point at infinity.

Lemma 2 *(Curtis [Cu1]) There are 132 such hexads (12 of type 1, 12 of type 2, 54 of type 3, and 54 of type 4). They form a Steiner system of type* $(5, 6, 12)$.

2 The MINIMOG description

Following Curtis' description [Cu2] of a Steiner system $S(5, 8, 24)$ using a 4×6 array, called the MOG, Conway [Co1] found and analogous description of $S(5, 6, 12)$ using a 3×4 array, called the MINIMOG. This section is devoted to the MINIMOG.

The *tetracode words* are

0	0	0	0	0	+	+	+	0	-	-	-
+	0	+	-	+	+	-	0	+	-	0	+
-	0	-	+	-	+	0	-	-	-	+	0

With "0″ = 0, "+″ = 1, "−″ = 2, these vectors form a linear code over $\mathbf{F_3}$. (This notation is Conway's. One must remember here that "+"+"+"="-" and "-"+"-"="+"!) They may also be described as the set of all 4-tuples in $\mathbf{F_3}$ of the form

$$(0, a, a, a), \quad (1, a, b, c), \quad (2, c, b, a),$$

where abc is any cyclic permutation of 012.

The *MINIMOG in the shuffle numbering* is the 3×4 array

$$\begin{array}{cccc} 6 & 3 & 0 & 9 \\ 5 & 2 & 7 & 10 \\ 4 & 1 & 8 & 11 \end{array}$$

We label the rows as follows:

- the first row has label 1,
- the second row has label +,
- the third row has label −.

$$\begin{array}{c|cccc|} \cline{2-5} 0 & 6 & 3 & 0 & 9 \\ + & 5 & 2 & 7 & 10 \\ - & 4 & 1 & 8 & 11 \\ \cline{2-5} \end{array}$$

A *col* (or column) is a placement of three + signs in a column of the array:

$$\begin{array}{|cccc|} \hline + & & & \\ + & & & \\ + & & & \\ \hline \end{array} \quad \begin{array}{|cccc|} \hline & + & & \\ & + & & \\ & + & & \\ \hline \end{array} \quad \begin{array}{|cccc|} \hline & & + & \\ & & + & \\ & & + & \\ \hline \end{array} \quad \begin{array}{|cccc|} \hline & & & + \\ & & & + \\ & & & + \\ \hline \end{array}$$

A *tet* (or tetrad) is a placement of 4 + signs having entries corresponding (as explained below) to a tetracode.

+	+	+	+	+				+			
					+	+	+				
									+	+	+
0	0	0	0	0	+	+	+	0	-	-	-

	+						+			+	
+		+		+	+			+			+
			+			+			+		
+	0	+	-	+	+	-	0	+	-	0	+

	+					+					+
			+		+					+	
+		+		+			+	+	+		
-	0	-	+	-	+	0	-	-	-	+	0

Each line in $\mathbf{F}_3^2$ with finite slope occurs once in the 3×3 part of some tet. The *odd man out* for a column is the label of the row corresponding to the non-zero digit in that column; if the column has no non-zero digit then the odd man out is a "?". Thus the tetracode words associated in this way to these patterns are the odd men out for the tets.

The *signed hexads* are the combinations 6-sets obtained from the MINIMOG from patterns of the form

col-col, col+tet, tet-tet, col+col-tet.

Lemma 3 *(Conway, [CS1], chapter 11, page 321) If we ignore signs, then from these signed hexads we get the 132 hexads of a Steiner system $S(5,6,12)$. These are all possible 6-sets in the shuffle labeling for which the odd men out form a part (in the sense that and odd man out "?" is ignored or regarded as a "wild-card") of a tetracode word and the column distribution is not $0,1,2,3$ in any order* [1].

Example 4 *Associated to the col-col pattern*

$$\begin{array}{|cccc|}\hline + & & & \\ + & & & \\ + & & & \\ \hline\end{array} \; - \; \begin{array}{|cccc|}\hline & + & & \\ & + & & \\ & + & & \\ \hline\end{array} \; = \; \begin{array}{|cccc|}\hline + & - & & \\ + & - & & \\ + & - & & \\ \hline\end{array}$$

[1]That is to say, the following cannot occur: some column has 0 entries, some column has exactly 1 entry, some column has exactly 2 entries, and some some column has exactly 3 entries.

is the tetracode 0 0 ? ? *and the signed hexad* $\{-1,-2,-3,4,5,6\}$ *and the hexad* $\{1,2,3,4,5,6\}$.

Associated to the col+tet pattern

$$\boxed{\begin{array}{cccc} & + & & \\ & + & & \\ & + & & \end{array}} - \boxed{\begin{array}{cccc} + & & & \\ & + & + & + \\ & & & \end{array}} = \boxed{\begin{array}{cccc} + & + & & \\ & - & + & + \\ & + & & \end{array}}$$

is the tetracode 0 + + + *and the signed hexad* $\{1,-2,3,6,7,10\}$ *and the hexad* $\{1,2,3,6,7,10\}$.

Furthermore, it is known [Co1] that the Steiner system $S(5,6,12)$ in the shuffle labeling has the following properties.

- There are 11 hexads with total 21 and none with lower total.
- The complement of any of these 11 hexads in $\{0,1,...,11\}$ is another hexad.
- There are 11 hexads with total 45 and none with higher total.

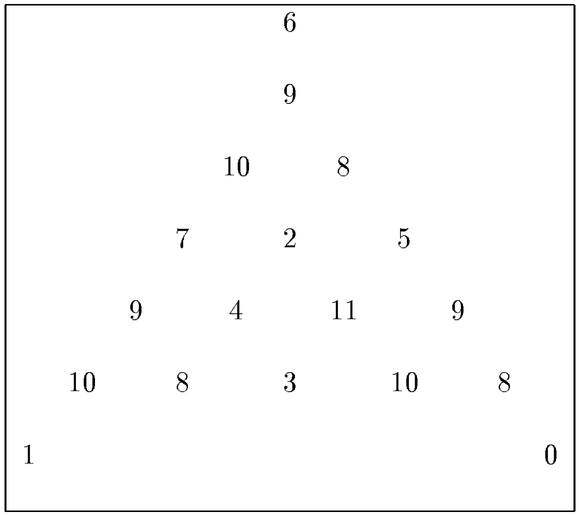
3 The shuffle kitten

Since Steiner systems $S(5,6,12)$ are unique up to relabelings, we should expect a "kitten" for the shuffle labeling. There is one and this section describes it.

In Conway, [Co1], the MINIMOG for the "modulo 11 labeling" is given:

$$\begin{array}{cccc} 0 & 3 & \infty & 2 \\ 5 & 9 & 8 & 10 \\ 4 & 1 & 6 & 7 \end{array}$$

Comparing this MINIMOG with that for the shuffle labeling, we obtain the following kitten.



The Shuffle Kitten.

The "views" from each of the three "points at infinity" is given in the following tables.

5	11	3	5	11	3	8	10	3
8	2	4	2	4	8	9	11	4
9	10	7	7	9	10	5	2	7
picture at 6			picture at 1			picture at 0		

Example 5
- *0,2,4,5,6,11 is a square in the picture at 1.*
- *0,2,3,4,5,7 is a cross in the picture at 0.*

4 Mathematical blackjack

Mathematical blackjack is a 2-person combinatorial game whose rules will be described below. What is remarkable about it is that a winning strategy, discovered by Conway and Ryba [CS2] and [KR], depends on knowing how to determine hexads in the Steiner system $S(5,6,12)$ using the shuffle labeling.

Winning ways in mathematical blackjack

Mathematical blackjack is played with 12 cards, labeled $0,...,11$ (for example: *king*, *ace*, 2, 3, ..., 10, *jack*, where the *king* is 0 and the *jack* is 11). Divide the 12 cards into two piles of 6 (to be fair, this should be done randomly). Each of the 6 cards of one of these piles are to be placed face up on the table. The remaining cards are in a stack which is shared and visible to both players. If the sum of the cards face up on the table is less than or equal to 21 then no legal move is possible so you must shuffle the cards and deal a new game. (Conway [Co2] calls such a game $0 = \{|\}$; in this game the first player automatically loses and so you courteously offered the first move!.)

- Players alternate moves.
- A move consists of exchanging a card on the table with a lower card from the other pile.
- The player whose move makes the sum of the cards on the table under 21 loses.

The winning strategy (given below) for this game is due to Conway and Ryba [CS2], [KR]. There is a Steiner system $S(5,6,12)$ of hexads in the set $\{0,1,...,11\}$. This Steiner system is associated to the MINIMOG of in the "shuffle numbering" rather than the "modulo 11 labeling".

Proposition 6 *(Ryba) For this Steiner system, the winning strategy is to choose a move which is a hexad from this system.*

This result is proven in [KR].

If you are unfortunate enough to be the first player starting with a hexad from $S(5,6,12)$ then, according to this strategy and properties of Steiner systems, there is no winning move. In a randomly dealt game there is a probability of

$$\frac{132}{\binom{12}{6}} = 1/7$$

that the first player will be dealt such a hexad, hence a losing position. In other words, we have the following result.

Lemma 7 *The probability that the first player has a win in mathematical blackjack (with a random initial deal) is* $6/7$.

Example 8
- *Initial deal: 0,2,4,6,7,11. The total is* 30. *The pattern for this deal is*

•		•	
	•	•	
•			•

where • is a ±. No combinations of ± choices will yield a tetracode odd men out, so this deal is not a hexad.

- *First player replaces 7 by 5: 0,2,4,5,6,11. The total is now 28. (Note this is a square in the picture at 1.) This corresponds to the col+tet*

+		+	
+	+		
-			+

  with tetracode odd men out $- \ + \ 0 \ -$*.*

- *Second player replaces 11 by 7: 0,2,4,5,6,7. The total is now 24. Interestingly, this 6-set corresponds to the pattern*

•		•	
•	•	•	
•			

  (hence possible with tetracode odd men out $0 + + \,?$*, for example). However, it has column distribution* $3, 1, 2, 0$*, so it cannot be a hexad.*

- *First player replaces 6 by 3: 0,2,3,4,5,7. (Note this is a cross in the picture at 0.) This corresponds to the tet-tet pattern*

	+	-	
+	-	+	
-			

  with tetracode odd men out $- \ - \ + \ 0$*. Cards total 21. First player wins.*

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